PSEUDO-BCH SEMILATTICES

Abstract

In this paper we study pseudo-BCH algebras which are semilattices or lattices with respect to the natural relation \( \leq \); we call them pseudo-BCH join-semilattices, pseudo-BCH meet-semilattices and pseudo-BCH lattices, respectively. We prove that the class of all pseudo-BCH join-semilattices is a variety and show that it is weakly regular, arithmetical at 1, and congruence distributive. In addition, we obtain the systems of identities definining pseudo-BCH meet-semilattices and pseudo-BCH lattices.

Keywords: (pseudo-)BCK/BCI/BCH algebra, pseudo-BCH join (meet)-semilattice, weakly regular, arithmetical at 1.

2010 Mathematics Subject Classification: 03G25, 06A12, 06F35

1. Introduction

In 1966, Imai and Iséki ([8, 11]) introduced BCK and BCI algebras as algebras connected to certain kinds of logics. In 1983, Hu and Li ([7]) defined BCH algebras. It is known that BCK and BCI algebras are contained in the class of BCH algebras. In [9, 10], Iorgulescu introduced many interesting generalizations of BCI or of BCK algebras.

In 2001, Georgescu and Iorgulescu ([6]) defined pseudo-BCK algebras as an extension of BCK algebras. In 2008, Dudek and Jun ([2]) introduced pseudo-BCI algebras as a natural generalization of BCI algebras and of pseudo-BCK algebras. These algebras have also connections with other algebras of logic such as pseudo-MV algebras and pseudo-BL algebras defined by Georgescu and Iorgulescu in [4] and [5], respectively. Recently, Walendziak ([14]) introduced pseudo-BCH algebras as an extension of BCH algebras.
In [13], Kühr investigated pseudo-BCK algebras whose underlying posets are semilattices. In this paper we study pseudo-BCH join-semilattices, that is, pseudo-BCH algebras which are join-semilattices with respect to the natural relation \( \leq \). We prove that the class of all pseudo-BCH join-semilattices is a variety and show that it is weakly regular, arithmetical at 1, and congruence distributive. In addition, we obtain the systems of identities defining pseudo-BCH meet-semilattices and pseudo-BCH lattices.

2. Preliminaries

We recall that an algebra \((X; \rightarrow, 1)\) of type \((2, 0)\) is called a BCH algebra if it satisfies the following axioms:

(BCH-1) \( x \rightarrow x = 1; \)
(BCH-2) \( x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z); \)
(BCH-3) \( x \rightarrow y = y \rightarrow x = 1 \implies x = y. \)

A BCI algebra is a BCH algebra \((X; \rightarrow, 1)\) satisfying the identity

(BCI) \( (y \rightarrow z) \rightarrow ((z \rightarrow x) \rightarrow (y \rightarrow x)) = 1. \)

A BCK algebra is a BCI algebra \((X; \rightarrow, 1)\) such that \( x \rightarrow 1 = 1 \) for all \( x \in X. \)

A pseudo-BCI algebra ([2]) is a structure \((X; \leq, \rightarrow, \leadsto, 1)\), where \( \leq \) is a binary relation on the set \( X \), \( \rightarrow \) and \( \leadsto \) are binary operations on \( X \) and 1 is an element of \( X \), verifying the axioms:

(pBCI-1) \( y \rightarrow z \leq (z \rightarrow x) \leadsto (y \rightarrow x), y \leadsto z \leq (z \leadsto x) \rightarrow (y \leadsto x); \)
(pBCI-2) \( x \leq (x \leadsto y) \rightarrow y, x \leq (x \rightarrow y) \leadsto y; \)
(pBCI-3) \( x \leq x; \)
(pBCI-4) \( x \leq y, y \leq x \implies x = y; \)
(pBCI-5) \( x \leq y \iff x \rightarrow y = 1 \iff x \leadsto y = 1. \)

A pseudo-BCI-algebra \((X; \leq, \rightarrow, \leadsto, 1)\) is called a pseudo-BCK algebra if it satisfies the identities

(pBCK) \( x \rightarrow 1 = x \leadsto 1 = 1. \)

Definition 2.1. ([14]) A (dual) pseudo-BCH algebra is an algebra \( \mathfrak{X} = (X; \rightarrow, \leadsto, 1) \) of type \((2, 2, 0)\) satisfying the axioms:

(pBCH-1) \( x \rightarrow x = x \leadsto x = 1; \)
(pBCH-2) \( x \rightarrow (y \leadsto z) = y \leadsto (x \rightarrow z); \)
(pBCH-3)  \[ x \rightarrow y = y \sim x = 1 \implies x = y; \]
(pBCH-4)  \[ x \rightarrow y = 1 \iff x \sim y = 1. \]

**Remark 2.2.** Observe that if \((X; \rightarrow, 1)\) is a BCH algebra, then letting \(x \rightarrow y := x \sim y\), produces a pseudo-BCH algebra \((X; \rightarrow, \sim, 1)\). Therefore, every BCH algebra is a pseudo-BCH algebra in a natural way. It is easy to see that if \((X; \rightarrow, \sim, 1)\) is a pseudo-BCH algebra, then \((X; \sim, \rightarrow, 1)\) is also a pseudo-BCH algebra. From Proposition 3.2 of [2] we conclude that if \((X; \leq, \rightarrow, \sim, 1)\) is a pseudo-BCI algebra, then \((X; \rightarrow, \sim, 1)\) is a pseudo-BCH algebra.

In any pseudo-BCH algebra we can define a natural relation \(\leq\) by putting
\[ x \leq y \iff x \rightarrow y = 1 \iff x \sim y = 1. \]

It is easy to see that \(\leq\) is reflexive and anti-symmetric but it is not transitive in general (see Example 2.3 below). We note that in pseudo-BCK/BCI algebras the relation \(\leq\) is a partial order.

**Example 2.3.** Let \(X = \{a, b, c, d, e, f, 1\}\). We define the binary operations \(\rightarrow\) and \(\sim\) on \(X\) as follows

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Then \(X = (X; \rightarrow, \sim, 1)\) is a pseudo-BCH algebra (see Example 2.6 of [15]). We have \(d \leq e\) and \(e \leq f\) but \(d \not\leq f\), and therefore \(\leq\) is not transitive.

**Proposition 2.4.** ([14]) Every pseudo-BCH algebra \(\mathfrak{X}\) satisfies, for all \(x, y \in X\), the following conditions:

(i)  \[ 1 \rightarrow x = 1 \sim x = x, \]
(ii) \[ x \leq (x \sim y) \rightarrow y, \text{ and } x \leq (x \rightarrow y) \sim y. \]
Proposition 2.5. ([14]) Let $X$ be a pseudo-BCH algebra. Then $X$ is a pseudo-BCI algebra if and only if it verifies the following implication: for all $x, y, z \in X$,

$$x \leq y \implies (z \to x \leq z \to y, z \rightsquigarrow x \leq z \rightsquigarrow y).$$

(2.1)

3. Pseudo-BCH semilattices

Generalizing the notion of a pseudo-BCK semilattice (see [13]) we define pseudo-BCH join-semilattices.

Definition 3.1. We say that an algebra $(X; \vee, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCH join-semilattice if $(X; \vee)$ is a join-semilattice, $(X; \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCH-algebra and $x \vee y = y \iff x \to y = 1$ for all $x, y \in X$.

Example 3.2. Let $X = \{a, b, c, 1\}$. We define the binary operations $\rightarrow$ and $\rightsquigarrow$ on $X$ as follows:

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It is easy to check that $(X; \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCH algebra. Since $X$ is a join-semilattice with respect to $\vee$ (under $\leq$), we conclude that $\mathcal{X} = (X; \vee, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCH join-semilattice; it is even a chain with $c < b < a < 1$.

Example 3.3. Let $\mathcal{X} = (\{a, b, c, d, e, f, 1\}; \rightarrow, \rightsquigarrow, 1)$ be the pseudo-BCH algebra from Example 2.3. Since the relation $\leq$ is not transitive, $X$ is not a join-semilattice with respect to $\leq$. Therefore it is not a pseudo-BCH join-semilattice.

Proposition 3.4. Let $(X; \vee, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCH join-semilattice. The following properties hold (for all $x, y, z \in X$):

(a1) $x \vee y = y \vee x$,
(a2) $(x \vee y) \vee z = x \vee (y \vee z)$,
(a3) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
\textbf{Proof:} (a1)–(a3) and (a5) are obvious. By Proposition 2.4 (i) we get (a4). Identities (a6) and (a7) follow from Proposition 2.4 (ii).

\textbf{Proposition 3.5.} Let \((X; \lor, \rightarrow, \leadsto, 1)\) be an algebra of type \((2, 2, 2, 0)\) satisfying (a1)–(a7). Define \(\leq\) on \(X\) by

\[ x \leq y \iff x \lor y = y. \]

Then, for all \(x, y, z \in X\), we have:

1. \(x \leq y \text{ and } y \leq x \text{ imply } x = y,\)
2. \(x \leq y \text{ and } y \leq z \text{ imply } x \leq z,\)
3. \(x \leq y \iff x \rightarrow y = 1,\)
4. \(x \leq y \iff x \leadsto y = 1,\)
5. \(x \lor 1 = 1 \lor x = 1 \text{ (that is, } x \leq 1),\)
6. \(x \rightarrow 1 = x \leadsto 1 = 1,\)
7. \(x \rightarrow x = x \leadsto x = 1 \text{ (that is, } x \leq x).\)

\textbf{Proof:} Statements (1) and (2) follow from (a1) and (a2), respectively.

To prove (3), let \(x, y \in X\) and \(x \lor y = y.\) Applying (a5), we get \(x \rightarrow y = 1.\)

Conversely, suppose that \(x \rightarrow y = 1.\) Hence \((x \rightarrow y) \leadsto y = 1 \leadsto y = y\) by (a4). From (a7) we see that \(x \lor y = y,\) that is, \(x \leq y.\)

(4) The proof of (4) is similar to that of (3).

(5) Applying (a5) and (a4), we obtain \(1 = 1 \rightarrow (1 \lor x) = 1 \lor x.\) This clearly forces (5).

(6) By (5), \(x \leq 1.\) Using (3) and (4), we get (6).

(7) We have

\[
1 = (1 \leadsto x) \rightarrow x \lor 1 \quad \text{[by (5)]}
\]

\[
= (1 \leadsto x) \rightarrow x \quad \text{[by (a6)]}
\]

\[
= x \rightarrow x. \quad \text{[by (a4)]}
\]

Similarly, \(x \leadsto x = 1.\)

\textbf{Combining} Propositions 3.4 and 3.5 we get
Theorem 3.6. An algebra \((X; \lor, \to, \leadsto, 1)\) of type \((2, 2, 2, 0)\) is a pseudo-BCH join-semilattice if and only if it satisfies the identities \((a1)-(a7)\).

From Proposition 3.5 (6) we have

Corollary 3.7. Every pseudo-BCH join-semilattice verifies \((pBCK)\).

Let us denote by \(J\) the class of all pseudo-BCH join-semilattices.

Remark 3.8. The class \(J\) is a variety. Therefore \(J\) is closed under the formation of homomorphic images, subalgebras, and direct products.

The disjoint union of BCK algebras was introduced by Iséki and Tanaka in [12] and next generalized to BCH algebras ([3]) and pseudo-BCH algebras ([15]). Below we extend this concept to the case of pseudo-BCH join-semilattices.

Let \(T\) be any set and, for each \(t \in T\), let \(X_t = (X_t; \lor_t, \to_t, \leadsto_t, 1)\) be a pseudo-BCH join-semilattice. Suppose that \(X_s \cap X_t = \{1\}\) for \(s, t \in T, s \neq t\). Set \(X = \bigcup_{t \in T} X_t\) and define the binary operations \(\lor, \to\) and \(\leadsto\) on \(X\) via

\[
x \lor y = \begin{cases} x \lor_t y & \text{if } x, y \in X_t, t \in T, \\ 0 & \text{if } x \in X_s, y \in X_t, s, t \in T, s \neq t. \end{cases}
\]

\[
x \to y = \begin{cases} x \to_t y & \text{if } x, y \in X_t, t \in T, \\ x & \text{if } x \in X_s, y \in X_t, s, t \in T, s \neq t. \end{cases}
\]

and

\[
x \leadsto y = \begin{cases} x \leadsto_t y & \text{if } x, y \in X_t, t \in T, \\ x & \text{if } x \in X_s, y \in X_t, s, t \in T, s \neq t. \end{cases}
\]

It is easily seen that \(X = (X; \lor, \to, \leadsto, 1)\) is a pseudo-BCH join-semilattice; it will be called the disjoint union of \((X_t)_{t \in T}\).

Example 3.9. Let \(X_1 = X\), where \(X = (\{a,b,c,1\}; \lor, \to, \leadsto, 1)\) is the pseudo-BCH join-semilattice from Example 3.2. Consider the set \(X_2 = \{d, e, f, 1\}\) with the operations \(\to_2\) and \(\lor_2\) defined by the following tables:

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Let $\rightsquigarrow_2 := \to_2$. Routine calculations show that $\mathcal{X}_2 = (X_2; \lor_2, \to_2, \rightsquigarrow_2, 1)$ is a (pseudo)-BCH join-semilattice. Let $X' = \{a, b, c, d, e, f, 1\}$. We define the binary operations $\to'$ and $\rightsquigarrow'$ on $X'$ as follows:

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It is clear that $\mathcal{X}' = (X'; \lor', \to', \rightsquigarrow', 1)$, where the operation $\lor'$ is illustrated in Figure 1, is the disjoint union of $\mathcal{X}_1$ and $\mathcal{X}_2$.

**Figure 1**

**Proposition 3.10.** Let $\mathcal{X} = (X; \lor, \to, \rightsquigarrow, 1)$ be a pseudo-BCH join-semilattice. Then the following statements are equivalent:

(i) $\mathcal{X}$ is a pseudo-BCK join-semilattice.

(ii) $\mathcal{X}$ satisfies (2.1) for all $x, y, z \in X$.

**Proof:** Follows immediately from Proposition 2.5 and Corollary 3.7. \(\Box\)

**Proposition 3.11.** Let $\mathcal{X} = (X; \lor, \to, \rightsquigarrow, 1)$ be a pseudo-BCH join-semilattice satisfying the following implication: for all $x, y, z \in X$,

$$x \leq y \implies (y \to x) \rightsquigarrow x = (y \rightsquigarrow x) \to x = y. \tag{3.1}$$

Then $\mathcal{X}$ is a pseudo-BCK join-semilattice.
Proof: Let \( x, y, z \in X \) and \( x \leq y \). By (pBCH-2), (pBCH-1) and (pBCK),

\[
(z \to x) \to (z \to y) = (z \to x) \to ((y \to x) \leftrightarrow x))
= (y \to x) \leftrightarrow ((z \to x) \to (z \to x))
= (y \to x) \leftrightarrow 1
= 1.
\]

Then \( z \to x \leq z \to y \). Similarly, \( z \leftrightarrow x \leq z \leftrightarrow y \). From Proposition 3.10 we see that \( X \) is a pseudo-BCK join-semilattice. \qed

Remark 3.12. The converse of Proposition 3.11 is false. Indeed, let \( X \) be the pseudo-BCH join-semilattice from Example 3.2. It is easy to check that \( X \) satisfies implication (2.1), and therefore it is a pseudo-BCK join-semilattice. However, (3.1) does not hold in \( X \), because we have \( c < a \) and \( (a \leftrightarrow c) \to c = 1 \).

Definition 3.13. An algebra \( (X; \land, \to, \leftrightarrow, 1) \) is called a pseudo-BCH meet-semilattice if \( (X; \land) \) is a meet-semilattice, \( (X; \to, \leftrightarrow, 1) \) is a pseudo-BCH algebra, and \( x \land y = x \iff x \to y = 1 \) for all \( x, y \in X \).

Denote by \( \mathcal{M} \) the class of all pseudo-BCH meet-semilattices.

Proposition 3.14. An algebra \( X = (X; \land, \to, \leftrightarrow, 1) \) of type \((2, 2, 2, 0)\) is a pseudo-BCH meet-semilattice if and only if it satisfies the following identities:

\[
\begin{align*}
(b1) & \quad x \land x = x, \\
(b2) & \quad x \land y = y \land x, \\
(b3) & \quad x \land (y \land z) = (x \land y) \land z, \\
(b4) & \quad x \to (y \leftrightarrow z) = y \leftrightarrow (x \to z), \\
(b5) & \quad 1 \to x = 1 \leftrightarrow x = x, \\
(b6) & \quad (x \land y) \to y = 1 = (x \land y) \leftrightarrow y, \\
(b7) & \quad x \land ((x \leftrightarrow y) \to y) = x = x \land ((x \to y) \leftrightarrow y).
\end{align*}
\]

Proof: Obviously, every pseudo-BCH meet-semilattice satisfies the axioms (b1)–(b7).

Conversely, let (b1)–(b7) hold in \( X \). Clearly, \( (X; \land) \) is a meet-semilattice. Define \( \leq \) on \( X \) by

\[
x \leq y \iff x = x \land y.
\]

Observe that

\[
x \leq y \iff x \to y = 1 \iff x \leftrightarrow y = 1 \tag{3.2}
\]
for all \( x, y \in X \). Let \( x \leq y \), that is, \( x \land y = x \). By (b6), \( x \to y = 1 \) and \( x \leadsto y = 1 \). Suppose now that \( x \to y = 1 \). Applying (b7) and (b5), we get

\[
x = x \land ((x \to y) \leadsto y) = x \land (1 \leadsto y) = x \land y.
\]

Hence \( x \leq y \). Similarly, if \( x \leadsto y = 1 \), then \( x \leq y \). Thus (3.2) holds. Therefore, we deduce that \((X; \to, \leadsto, 1)\) is a pseudo-BCH algebra, and finally that \((X; \land, \to, \leadsto, 1)\) is a pseudo-BCH meet-semilattice.

\[\Box\]

**Corollary 3.15.** The class \( \mathcal{M} \) is a variety.

**Definition 3.16.** An algebra \((X; \lor, \land, \to, \leadsto, 1)\) is called a pseudo-BCH lattice if \((X; \lor, \land)\) is a lattice, \((X; \to, \leadsto, 1)\) is a pseudo-BCH algebra, and \( x \to y = 1 \iff x \lor y = y \iff x \land y = x \) for all \( x, y \in X \).

Denote by \( \mathcal{L} \) the class of all pseudo-BCH lattices.

**Example 3.17.** Let \( X = \{a, b, c, d, 1\} \). Define binary operations \( \to \) and \( \leadsto \) on \( X \) by the following tables:

<table>
<thead>
<tr>
<th>( \to )</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>1</th>
<th>( \leadsto )</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>1</td>
<td>a</td>
<td>1</td>
<td>b</td>
<td>b</td>
<td>d</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>1</td>
<td>a</td>
<td>d</td>
<td>1</td>
<td>b</td>
<td>a</td>
<td>1</td>
<td>a</td>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>b</td>
<td>1</td>
<td>c</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>d</td>
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<td>1</td>
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</tr>
<tr>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>1</td>
</tr>
</tbody>
</table>

By routine calculation, \( X = (X; \to, \leadsto, 1) \) is a pseudo-BCH algebra. We shall represent the set \( X \) and the binary relation \( \leq \) by the following Hasse diagram:
Therefore, \((X; \lor, \land, \rightarrow, \rightsquigarrow, 1)\) is a pseudo-BCH lattice.

Remark 3.18. The class \(\mathcal{L}\) is a variety that is axiomatized by the defining identities of lattices and by the identities \((a3)\)–\((a7)\) or by \((b4)\)–\((b7)\), respectively.

Now we recall several universal algebraic notions (see e. g. [1]). We will denote by \(\text{Con}\mathfrak{A}\) the congruence lattice of an algebra \(\mathfrak{A}\). For \(\theta \in \text{Con}\mathfrak{A}\) and \(x \in A\), let \(x/\theta\) denote the equivalence class of \(x\) modulo \(\theta\). An algebra \(\mathfrak{A}\) with a constant 1 is called:

- weakly regular (at 1) if \(1/\theta = 1/\phi\) implies \(\theta = \phi\), for all \(\theta, \phi \in \text{Con}\mathfrak{A}\);
- permutative at 1 if \(1/(\theta \circ \phi) = 1/(\phi \circ \theta)\) for all \(\theta, \phi \in \text{Con}\mathfrak{A}\);
- distributive at 1 if \(1/\theta \cap (\phi \lor \psi) = 1/(\theta \cap \phi) \lor (\theta \cap \psi)\) for all \(\theta, \phi, \psi \in \text{Con}\mathfrak{A}\);
- arithmetical at 1 if it is both permutative at 1 and distributive at 1.

Let \(\mathcal{V}\) be a variety of algebras with a constant 1. We say that \(\mathcal{V}\) is weakly regular (resp., permutative at 1, distributive at 1, and arithmetical at 1) if every algebra \(\mathfrak{A} \in \mathcal{V}\) is weakly regular (resp., permutative at 1, distributive at 1, and arithmetical at 1). It is known that a variety \(\mathcal{V}\) is weakly regular if and only if there exist binary terms \(t_1, \ldots, t_n\) for some \(n \in \mathbb{N}\) such that

\[
t_1(x, y) = \cdots = t_n(x, y) = 1 \iff x = y.
\] (3.3)

A variety is arithmetical at 1 if and only if there exists a binary term \(t\) satisfying \(t(x, x) = t(1, x) = 1\) and \(t(x, 1) = x\). A variety \(\mathcal{V}\) is congruence distributive if \(\text{Con}\mathfrak{A}\) is a distributive lattice for every \(\mathfrak{A} \in \mathcal{V}\).

Theorem 3.19. The variety \(\mathcal{J}\), \(\mathcal{M}\) and \(\mathcal{L}\) are weakly regular. Moreover, \(\mathcal{J}\) and \(\mathcal{L}\) are arithmetical at 1 and congruence distributive.

Proof: \(\mathcal{J}\), \(\mathcal{M}\) and \(\mathcal{L}\) are weakly regular since the terms \(t_1(x, y) = x \rightarrow y\) and \(t_2(x, y) = y \rightsquigarrow x\) satisfy (3.3) for \(n = 2\).

Let \(\mathfrak{X}\) be a pseudo-BCH join-semilattice and \(t(x, y) = y \rightarrow x\). Clearly, \(t(x, x) = 1\) and \(t(x, 1) = x\). By Corollary 3.7, \(\mathfrak{X}\) satisfies (pBCK), and hence \(t(1, x) = 1\). Then \(\mathfrak{X}\) is arithmetical at 1, and consequently distributive at 1.

Let \(\theta, \phi, \psi \in \text{Con}\mathfrak{X}\). By distributivity at 1, \(1/\theta \cap (\phi \lor \psi) = 1/(\theta \cap \phi) \lor (\theta \cap \psi)\). From weak regularity we obtain \(\theta \cap (\phi \lor \psi) = (\theta \cap \phi) \lor (\theta \cap \psi)\). Therefore \(\text{Con}\mathfrak{X}\) is a distributive lattice.
Thus pseudo-BCH join-semilattices (and hence pseudo-BCH lattices)
are arithmetical at 1 and congruence distributive.

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